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Chain conditions in amalgamated algebras along an ideal

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Abstract Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this paper, we study when the amalgamation of A with B along J with respect to f is a ϕ-ring. Hence, we study two different chain conditions over this structure. Namely, the nonnil-Noetherian condition and the Noetherian spectrum condition.

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الملخص

لتكن A وَ B حلقتين وَ J مثالياً في B وَ $f : A \to B$ تشاكلَ حلقات. في هذه الورقة، ندرس متى يكون دمج A مع B بواسطة J بالنسبة لِـ f حلقة-ϕ. لذا ندرس نوعين مختلفين من شروط السلسلة على هذا التركيب وهما: الشرط النوثيري على المثاليات غير المحتواة في الأساس المعدَم وشرط الطيف النوثيري.

1 Introduction

Throughout this paper, all rings are commutative with unity. We denote by Nilp (R) the set of nilpotent elements of the ring R. By (a) we denote the ideal of R generated by $a \in R$.

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A with B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [11] and [12]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [8], [9] and [10]). Moreover, other classical

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constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation [11, Examples 2.5 and 2.6] and other classical constructions, such as the Nagata's idealization, cf. [17, page 2], and the CPI extensions (in the sense of Boisen and Sheldon [7]) are strictly related to it [11, Example 2.7 and Remark 2.8].

On the other hand, the amalgamation $A \bowtie^f J$ is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [14], concerning the embedding of a ring without identity in a ring with identity. An ample introduction on the genesis of the notion of amalgamation is given in [11, Section 2]. Also, the authors consider the iteration of the amalgamation process, giving some geometrical applications of it.

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [11, Section 4]. This point of view allows the authors in [11] and [12] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [11], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [12], they pursue the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Recall from [3] and [13] that a prime ideal of R is called a *divided prime ideal* if $P \subseteq (x)$ for every $x \in R \setminus P$; thus a divided prime ideal is comparable to every ideal of R. In [2], [4] and [5], the author paid attention to the class of rings

$$\mathcal{H} = \{R \mid R \text{ is a commutative ring and } \mathrm{Nilp}\,(R) \text{is a divided prime ideal of} R\}$$

Observe that if R is an integral domain, then $R \in \mathcal{H}$. If $R \in \mathcal{H}$, then R is called a *ϕ-ring*.

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this paper, we study when the amalgamation of A with B along J with respect to f is a ϕ-ring.

Recall that a ring R is said to be *nonnil-Noetherian* if each ideal of R which is not contained in the nilradical of R is finitely generated. The treatment of this notion in the context of the class of ϕ-rings was established in [6], where the author proved that many of the properties of Noetherian rings are true for nonnil-Noetherian rings. Trivially, Noetherian rings are nonnil-Noetherian but the converse is not true in general, cf. [6, Theorem 3.4]. In Sect. 2, we characterize when $A \bowtie^f J$ is nonnil-Noetherian provided it is ϕ-ring. Recall that a ring R has *Noetherian spectrum* if it satisfies the ascending chain condition for radical ideals. Every nonnil-Noetherian ring has Noetherian spectrum and the converse is false; cf. [16, Proposition 1.8 and Remark 1.9]. In Sect. 2, we characterize when $A \bowtie^f J$ is of Noetherian spectrum.

2 Main results

We begin with the following result in which we study when $A \bowtie^f J$ is a ϕ-ring.

Theorem 2.1 *Let A and B be two rings, J be an ideal of B and let $f : A \to B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-ring then the following properties hold:*

1. *A is a ϕ-ring.*
2. *if J is a prime ideal of $f(A)+J$ or $f^{-1}(J) \subseteq \mathrm{Nilp}\,(A)$ then $f(A)+J$ is a ϕ-ring.*

Conversely, assume that $J = \mathrm{Nilp}\,(B)$ and that $f^{-1}(J) \subseteq \mathrm{Nilp}\,(A)$ then if $f(A)+J$ and A are ϕ-rings then so is $A \bowtie^f J$.

Proof Clearly, $\mathrm{Nilp}\,(A \bowtie^f J) = \{(a, f(a)+j) \mid a \in \mathrm{Nilp}\,(A),\ j \in \mathrm{Nilp}\,(B) \cap J\}$.

Assume that $A \bowtie^f J$ is a ϕ-ring. Then, $\mathrm{Nilp}\,(A \bowtie^f J)$ is a prime ideal of $A \bowtie^f J$. From [12, Proposition 2.6], there are two possible cases:

Case 1: There exists a prime ideal P of A such that

$$\mathrm{Nilp}\,(A \bowtie^f J) = P \bowtie^f J := \{(p, f(p)+j) \mid p \in P,\ j \in J\}$$

Then, $\mathrm{Nilp}\,(A) = P$, and so it is prime. Consider $a \notin \mathrm{Nilp}\,(A)$. Then, $(a, f(a)) \notin \mathrm{Nilp}\,(A \bowtie^f J)$. Hence, $\mathrm{Nilp}\,(A \bowtie^f J) \subseteq (a, f(a))A \bowtie^f J$ since $\mathrm{Nilp}\,(A \bowtie I)$ is a divided prime ideal of $A \bowtie^f J$. Thus, for each $x \in \mathrm{Nilp}\,(A)$, there exists $(b, f(b)+j) \in A \bowtie^f J$ such that $(x, f(x)) = (b, f(b)+j)(a, f(a))$. Hence,

$x = ba$, and so Nilp $(A) \subseteq aA$. Thus, Nilp (A) is a divided prime ideal of A. Consequently, A is a ϕ-ring. Moreover, $\{0\} \times J \subseteq P \bowtie^f J = \text{Nilp}\,(A \bowtie^f J)$. Hence, $J \subseteq \text{Nilp}\,(f(A) + J)$.

Case 2: There exists a prime ideal Q of B with $J \not\subseteq Q$ such that

$$\text{Nilp}\,(A \bowtie^f J) = \overline{Q}^f := \{(a, f(a) + j) \mid a \in A,\ j \in J,\ f(a) + j \in Q\}$$

Let $j \in J \backslash Q$. Then, $(0, j) \notin \overline{Q}^f$. Thus, Nilp $(A \bowtie^f J) = \overline{Q}^f \subseteq (0, j)A \bowtie^f J \subseteq \{0\} \times J$ since Nilp $(A \bowtie^f J)$ is a divided prime ideal of $A \bowtie^f J$. Hence, Nilp $(A) = \{0\}$. Let $x, y \in A$ such that $xy = 0$. Then, $(x, f(x))(y, f(y)) = (0, 0) \in \overline{Q}^f$. Hence, $(x, f(x)) \in \overline{Q}^f$ or $(y, f(y)) \in \overline{Q}^f$. Thus, $x = 0$ or $y = 0$. Consequently, A is an integral domain. So, A is a ϕ-ring.

Assume that J is a prime ideal of $f(A) + J$. Let $(f(a) + j)(f(b) + j') \in \text{Nilp}\,(f(A) + J)$. There exists a positive integer k such that $[(f(a) + j)(f(b) + j')]^k = 0$. Hence, $f(ab)^k \in J$. Thus, $f(a) \in J$ or $f(b) \in J$. Suppose that $f(a) \in J$. Then, $(0, f(a) + j), (0, (f(a) + j)(f(b) + j')) \in A \bowtie^f J$, and we have $(0, (f(a)+j)(f(b)+j')) = (0, f(a)+j)(b, f(b)+j) \in \text{Nilp}\,(A \bowtie^f J)$. Then, $(0, f(a)+j) \in \text{Nilp}\,(A \bowtie^f J)$ or $(b, f(b) + j') \in \text{Nilp}\,(A \bowtie^f J)$. Hence, $f(a) + j \in \text{Nilp}\,(f(A) + J)$ or $f(b) + j \in \text{Nilp}\,(f(A) + J)$. Accordingly, Nilp $(f(A) + J)$ is a prime ideal of $f(A)+J$. Let $f(a)+j \notin \text{Nilp}\,(f(A)+J)$. Then, $(a, f(a)+j) \notin \text{Nilp}\,(A \bowtie^f J)$. Hence, Nilp $(A \bowtie^f J) \subseteq (a, f(a) + j)A \bowtie^f J$. Let $f(b) + j' \in \text{Nilp}\,(f(A) + J)$. There exists a positive integer k such that $(f(b) + j')^k = 0$. Hence, $f(b)^k \in J$. Then $f(b) \in J$ since J is a prime ideal of $f(A) + J$. Thus, $(0, f(b) + j') \in \text{Nilp}\,(A \bowtie^f J)$. Hence, there exists $(c, f(c) + j'') \in A \bowtie^f J$ such that $(0, f(b) + j') = (a, f(a) + j)(c, f(c) + j'')$. So, $f(b) + j' = (f(a) + j)(f(c) + j'')$. Thus, Nilp $(f(A) + J) \subseteq (f(a) + j)(f(A) + J)$. Hence, Nilp $(f(A) + J)$ is a divided prime ideal and so $f(A) + J$ is a ϕ-ring.

Assume that $f^{-1}(J) \subseteq \text{Nilp}\,(A)$. Let $(f(a) + j)(f(b) + j') \in \text{Nilp}\,(f(A) + J)$. There exists a positive integer k such that $[(f(a) + j)(f(b) + j')]^k = 0$. Hence, $[f(ab)]^k \in J$. Thus, $(ab)^k \in \text{Nilp}\,(A)$. Hence, $ab \in \text{Nilp}\,(A)$. So, $(a, f(a) + j)(b, f(b) + j') = (ab, (f(a) + j)(f(b) + j') \in \text{Nilp}\,(A \bowtie^f J)$. Then, $(a, f(a) + j) \in \text{Nilp}\,(A \bowtie^f J)$ or $(b, f(b) + j') \in \text{Nilp}\,(A \bowtie^f J)$. Hence, $f(a) + j \in \text{Nilp}\,(f(A) + J)$ or $f(b) + j \in \text{Nilp}\,(f(A) + J)$. Accordingly, Nilp $(f(A) + J)$ is a prime ideal of $f(A) + J$. Let $f(a) + j \notin \text{Nilp}\,(f(A)+J)$. Then, $(a, f(a)+j) \notin \text{Nilp}\,(A \bowtie^f J)$. Hence, Nilp $(A \bowtie^f J) \subseteq (a, f(a)+j)A \bowtie^f J$. Let $f(b)+j' \in \text{Nilp}\,(f(A) + J)$. There exists a positive integer k such that $(f(b)+j')^k = 0$. Hence, $f(b)^k \in J$. Then $b^k \in \text{Nilp}\,(A)$. So, $b \in \text{Nilp}\,(A)$. Thus, $(b, f(b)+j') \in \text{Nilp}\,(A \bowtie^f J)$. So, there exists $(c, f(c)+j'') \in A \bowtie^f J$ such that $(b, f(b) + j') = (a, f(a) + j)(c, f(c) + j'')$. So, $f(b) + j' = (f(a) + j)(f(c) + j'')$. Thus, Nilp $(f(A) + J) \subseteq (f(a) + j)(f(A) + J)$. Hence, Nilp $(f(A) + J)$ is a divided prime ideal and so $f(A) + J$ is a ϕ-ring.

Conversely, assume that $J = \text{Nilp}\,(B)$, $f^{-1}(J) \subseteq \text{Nilp}\,(A)$ and that $f(A)+J$ and A are ϕ-rings. Clearly, we have $J \subseteq \text{Nilp}\,(f(A) + J) \subseteq \text{Nilp}\,(B) = J$. Then, $J = \text{Nilp}\,(f(A) + J)$ and it is a prime ideal of $f(A) + J$. Let $(x, f(x) + j)(y, f(y) + j') \in \text{Nilp}\,(A \bowtie^f J)$. Then, $(f(x) + j)(f(y) + j') \in \text{Nilp}\,(f(A) + J)$. Hence, $(f(x) + j) \in \text{Nilp}\,(f(A) + J)$ or $(f(y) + j') \in \text{Nilp}\,(f(A) + J)$. Assume, for example, that $(f(x)+j) \in \text{Nilp}\,(f(A)+J)$ then there exists an integer k such that $(f(x)+j)^k = 0$. Then, $f(x)^k \in J$, which implies that $x^k \in \text{Nilp}\,(A)$. So, $x \in \text{Nilp}\,(A)$. Hence, $(x, f(x) + j) \in \text{Nilp}\,(A \bowtie^f J)$. Thus, Nilp $(A \bowtie^f J)$ is a prime ideal of $A \bowtie^f J$. Let $(x, f(x) + j) \notin \text{Nilp}\,(A \bowtie^f J)$. Then, $x \notin \text{Nilp}\,(A)$ (otherwise, $f(x) + j$ is also a nilpotent element, a contradiction). Thus, Nilp $(A) \subseteq xA$. If $f(x) + j \in \text{Nilp}\,(f(A) + J)$ then we can prove as above that $x \in \text{Nilp}\,(A)$, a contradiction. So, Nilp $(f(A) + J) \subseteq (f(x) + j)(f(A) + J)$. Consider $(a, f(a) + j') \in \text{Nilp}\,(A \bowtie^f J)$. There exists $\alpha \in A$ and $f(\beta) + j'' \in f(A) + J$ such that $a = \alpha x$ and $f(a)+j' = (f(\beta)+j'')(f(x)+j)$. Then, we have $(a, f(a)+j') = (\alpha, f(\beta)+j'')(x, f(x)+j)$. Moreover, it is easy to see that $f(\alpha)f(x) - f(\beta)f(x) \in J$. If $f(x) \in J$ then $x \in \text{Nilp}\,(A)$ which is impossible. Then, $(f(\alpha) - f(\beta)) \in J$. Thus, $(\alpha, f(\beta) + j'') \in A \bowtie^f J$. Then, Nilp $(A \bowtie^f J) \subseteq (x, f(x) + j)A \bowtie^f J$. Consequently, Nilp $(A \bowtie^f J)$ is a divided prime ideal. Thus, $A \bowtie^f J$ is a ϕ-ring. □

Remark 2.2 The assumption "J is a prime ideal of $f(A) + J$ or $f^{-1}(J) \subseteq \text{Nilp}\,(A)$" is necessary to show that $f(A) + J$ is a ϕ-ring. Consider the homomorphism of rings $f : \mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$; $n \mapsto \overline{n}$ and set $J = \{\overline{0}\}$ the zero ideal of $\mathbb{Z}/6\mathbb{Z}$. Clearly, $\mathbb{Z}$ is a ϕ-ring (since it is an integral domain). Moreover, $A \bowtie^f J \cong A$. Then, $A \bowtie^f J$ is also a ϕ-ring. But, $f(\mathbb{Z}) + J = \mathbb{Z}/6\mathbb{Z}$ is not a ϕ-ring. Indeed, Nilp $(\mathbb{Z}/6\mathbb{Z}) = \{\overline{0}\} = J$ which in not a prime ideal of $\mathbb{Z}/6\mathbb{Z}$ since $\overline{2} \times \overline{3} = \overline{6} = \overline{0} \in \text{Nilp}\,(\mathbb{Z}/6\mathbb{Z})$. We can see also that $f^{-1}(J) = 6\mathbb{Z} \not\subset \text{Nilp}\,(\mathbb{Z}) = \{0\}$.

Recall that if $A = B$, $f = \text{id}_A$ and J is an ideal of A, the ring $A \bowtie^{\text{id}_A} J$ coincides with the amalgamated duplication of A along the ideal J defined in [10], as follows:

$$A \bowtie J = \{(a, a + j) \mid a \in A,\ j \in J\}$$

Corollary 2.3 *Let A and B be two rings and let $f : A \to B$ be a ring homomorphism and assume that $f^{-1}(\mathrm{Nilp}\,(B)) \subseteq \mathrm{Nilp}\,(A)$. Then, $A \bowtie^f \mathrm{Nilp}\,(B)$ is a ϕ-ring if and only if A and $f(A) + \mathrm{Nilp}\,(B)$ are ϕ-rings.*

In particular, for each ring A, $A \bowtie \mathrm{Nilp}\,(A)$ is a ϕ-ring if and only if A is a ϕ-ring.

Proof The general case follows immediately from Theorem 2.1, while the particular case follows from the general one when $A = B$ and $f = \mathrm{id}_A$. □

Using Corollary 2.3, we can construct a new family of ϕ-rings.

Example 2.4 Let A be a ϕ-ring which is not integral domain. Set $A_1 = A \bowtie \mathrm{Nilp}\,(A)$ and for each $i \geq 1$ set $A_{i+1} = A_i \bowtie \mathrm{Nilp}\,(A_i)$. Then, $\{A_i\}_{i\geq 1}$ is a family of a ϕ-rings which are not integral domains.

Proof The fact that A_i is a ϕ-ring for each $i \geq 1$ follows from Corollary 2.3. If A_i is an integral domain for some $i \geq 1$ then by induction and by [11, Remark 5.3.], A is an integral domain, a contradiction. □

Proposition 2.5 *Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. If $A \bowtie^f J$ is a nonnil-Noetherian ring then so are A and $f(A) + J$.*

Proof By [16, Proposition 1.3], every homomorphic image of a nonnil-Noetherian ring is nonnil-Noetherian. Thus, if $A \bowtie^f J$ is nonnil-Noetherian, then so are A and $f(A) + J$ (by [11, Proposition 5.1(3)]). □

Remark 2.6 Let A and B be two rings, J an ideal of B and let $f : A \to B$ be a ring homomorphism. Set $\overline{A} = A/\mathrm{Nilp}\,(A)$, $\overline{B} = B/\mathrm{Nilp}\,(B)$, $\pi : B \to \overline{B}$ the canonical projection, and $\overline{J} = \pi(J)$. Consider the ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ defined by setting $\overline{f}(\overline{a}) = \overline{f(a)}$. It is easy to see that $\overline{f}$ is well defined and it is clearly a ring homomorphism. The kernel of the restriction to $A \bowtie^f J$ of the canonical projection $A \times B \to \overline{A} \times \overline{B}$ is obviously $\mathrm{Nilp}\,(A \bowtie^f J)$ and the image is $\overline{A} \bowtie^{\overline{f}} \overline{J}$. Hence, we have the following isomorphism of rings:

$$\begin{aligned} \psi : (A \bowtie^f J)/\mathrm{Nilp}\,(A \bowtie^f J) &\to \overline{A} \bowtie^{\overline{f}} \overline{J} \\ \overline{(a, f(a) + j)} &\mapsto (\overline{a}, \overline{f}(\overline{a}) + \overline{j}) \end{aligned}$$

Moreover, the rings $\overline{f}(\overline{A}) + \overline{J}$ and $(f(A) + J)/\mathrm{Nilp}\,(f(A) + J)$ are always isomorphic. Indeed, if λ is the restriction to $f(A)+J$ of the projection $B \to \overline{B}$, then clearly $\mathrm{Im}(\lambda) = \overline{f}(\overline{A})+\overline{J}$ and $\ker(\lambda) = \mathrm{Nilp}\,(f(A)+J)$.

In what follows we characterize $A \bowtie^f J$ to be nonnil-Noetherian under the assumption that it is a ϕ-ring.

Theorem 2.7 *Let A and B be two rings, $J \neq \{0\}$ an ideal of B and let $f : A \to B$ be a ring homomorphism. If $A \bowtie^f J$ is a ϕ-ring, the following are equivalent:*

1. *$A \bowtie^f J$ is a nonnil-Noetherian ring.*
2. *A and $f(A) + J$ are nonnil-Noetherian rings and $f^{-1}(J) \subseteq \mathrm{Nilp}\,(A)$.*

Proof (1) ⇒ (2) By Proposition 2.5, we have only to prove that $f^{-1}(J) \subseteq \mathrm{Nilp}\,(A)$. By [6, Theorem 2.2], $(A \bowtie^f J)/(\mathrm{Nilp}\,(A \bowtie^f J)$ is a Noetherian domain. Thus, by Remark 2.6, $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is a Noetherian domain.

If $\overline{J} = \{\overline{0}\}$, then, $J \subseteq \mathrm{Nilp}\,(B)$. Thus,

$$\mathrm{Nilp}\,(A \bowtie^f J) = \{(a, f(a) + j) \mid a \in \mathrm{Nilp}\,(A), j \in J\} = \mathrm{Nilp}\,(A) \bowtie^f J$$

Let $x \in f^{-1}(J)$. If $\mathrm{Nilp}\,(A \bowtie^f J) \subseteq (x, 0)A \bowtie^f J$, then $J = \{0\}$, which is impossible. Then, $(x, 0)A \bowtie^f J \subseteq \mathrm{Nilp}\,(A \bowtie^f J)$. Thus, $x \in \mathrm{Nilp}\,(A)$.

If $\overline{J} \neq \{\overline{0}\}$ then, by [11, Proposition 5.2], $\bar{f}^{-1}(\overline{J}) = 0$. Consequently, $f^{-1}(J) \subseteq \mathrm{Nilp}\,(A)$.

(2) ⇒ (1) Let $\bar{x} \in \bar{f}^{-1}(\overline{J})$. Then, $\overline{f(x)} = \bar{f}(\bar{x}) \in \overline{J}$. So, there exists an element $j \in J$ such that $(f(x)-j) \in \mathrm{Nilp}\,(B)$. Hence, there is an integer k such that $(f(x)-j)^k = 0$. Thus, $f(x^k) \in J$. Consequently, $x^k \in \mathrm{Nilp}\,(A)$. Thus, $x \in \mathrm{Nilp}\,(A)$, and $\bar{x} = 0$. Hence, $\bar{f}^{-1}(\overline{J}) = \{\overline{0}\}$. On the other hand, by Theorem 2.1, A and $f(A) + J$ are ϕ-rings. Thus, [6, Theorem 2.2], $\overline{A}$ and $(f(A) + J)/\mathrm{Nilp}\,(f(A) + J) \cong \overline{f}(\overline{A}) + \overline{J}$ are Noetherian domains. Hence, by [11, Proposition 5.6], $(\overline{A} \bowtie^{\overline{f}} \overline{J})$ is a Noetherian ring. Moreover, $\overline{A} \bowtie^{\overline{f}} \overline{J} \simeq (A \bowtie^f J)/\mathrm{Nilp}\,(A \bowtie^f J)$ which is an integral domain as $A \bowtie^f J$ is a ϕ-ring □

Corollary 2.8 *Let A be a ϕ-ring. Then, $A \bowtie \mathrm{Nilp}(A)$ is nonnil-Noetherian if and only if A is nonnil-Noetherian.*

Proof Follows from Theorem 2.7 and Corollary 2.3.

If J is a finitely generated A-module with the structure naturally induced by f, and J is a nonnil ideal of B, then the Noetherian and nonnil-Noetherian conditions coincide over the amalgamation of A with B along J with respect to f.

Proposition 2.9 *Assume that $J \not\subseteq \mathrm{Nilp}(B)$ and at least one of the following conditions holds:*

1. *J is a finitely generated A-module (with the structure naturally induced by f).*
2. *f is a finite homomorphism.*

Then, $A \bowtie^f J$ is nonnil-Noetherian if and only if $A \bowtie^f J$ is Noetherian.

Proof From [11, Proposition 5.7], under one of the conditions made above, $A \bowtie^f J$ is Noetherian if and only if A is Noetherian. If $A \bowtie^f J$ is Noetherian then clearly it is nonnil-Noetherian. Conversely, assume that $A \bowtie^f J$ is nonnil-Noetherian. Let P be a prime ideal of A. Then, $P \bowtie^f J$ is a prime ideal of $A \bowtie^f J$. Moreover, $P \bowtie^f J \not\subseteq \mathrm{Nilp}(A \bowtie^f J)$ since $J \not\subseteq \mathrm{Nilp}(B)$. Thus, $P \bowtie^f J$ is a finitely generated ideal of $A \bowtie^f J$. Hence, P is a finitely generated ideal of A. Thus, every prime ideal of A is finitely generated. So, A is Noetherian. Consequently, $A \bowtie^f J$ is Noetherian. □

In what follows, we give an example of a ring homomorphism $f : A \to B$ and an ideal J of B such that A and $f(A) + J$ are nonnil-Noetherian rings and $A \bowtie^f J$ is not nonnil-Noetherian.

We recall this construction. For a ring R, let B be an R-module. Consider

$$R(+)B = \{(r, b) \mid r \in R \text{ and } b \in B\}$$

and let (r, b) and (s, c) two elements of $R(+)B$. Define:

1. $(r, b) = (s, c)$ if $r = s$ and $b = c$.
2. $(r, b) + (s, c) = (r + s, b + c)$.
3. $(r, b)(s, c) = (rs, rc + sb)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity called the *Nagata's idealization of B in R*.

Example 2.10 Set $A = \mathbb{Z}(+)\mathbb{Q}$ and consider the surjective ring homomorphism $f : A \to \mathbb{Z}/6\mathbb{Z}$; $f((n, q)) = \overline{n}$. Consider $J = 3\mathbb{Z}/6\mathbb{Z} = \{\overline{0}, \overline{3}\}$ the ideal of $\mathbb{Z}/6\mathbb{Z}$. Then, A and $f(A) + J$ are nonnil-Noetherian rings. However, $A \bowtie^f J$ is not.

Proof By [6, Theorem 3.4], A is a nonnil-Noetherian ring which is not a Noetherian ring. On the other hand, $f(A) + J = \mathbb{Z}/6\mathbb{Z}$ is a Noetherian ring, and so a nonnil-Noetherian ring with $\mathrm{Nilp}(\mathbb{Z}/6\mathbb{Z}) = \{\overline{0}\}$. Moreover, $J \not\subseteq \mathrm{Nilp}(\mathbb{Z}/6\mathbb{Z})$ and J is a finitely generated A-module (with the structure naturally induced by f) since $J = \overline{3}\mathbb{Z}/6\mathbb{Z} = \overline{3}f(A) = \overline{3}.A$. If we suppose that $A \bowtie^f J$ is nonnil-Noetherian, then by Proposition 2.9, $A \bowtie^f J$ is Noetherian, and so is A, which is impossible. □

We end this paper with a characterization of $A \bowtie^f J$ to be of Noetherian spectrum.

Proposition 2.11 *The ring $A \bowtie^f J$ has Noetherian spectrum if and only A and $f(A) + J$ have Noetherian spectrum.*

In particular, if B has Noetherian spectrum, then $A \bowtie^f J$ has Noetherian spectrum if and only A has Noetherian spectrum.

Proof The result follows immediately by applying [15, Corollary 1.6], keeping in mind the fiber product structure of $A \bowtie^f J = \pi \circ f \times \pi$ where π is the canonical surjection $f(A) + J \to (f(A) + J)/J$, the fact that $(f(A) + J)/J$ is isomorphic to $A \bowtie^f J/f^{-1}(J) \times J$ and the fact that every subspace of a Noetherian topological space is still Noetherian.

In the particular case, $A \bowtie^f J = \pi_1 \circ f \times \pi_1$ where π_1 is the canonical surjection, $\pi_1 : B \to B/J$. □

We have the following consequence of the previous proposition.

Corollary 2.12 *Let A be a ring and I an ideal of A. Then, $A \bowtie I$ has Noetherian spectrum if and only if A has Noetherian spectrum.*

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References

1. Anderson, D.D.: Commutative rings. In: Multiplicative Ideal Theory in Commutative Algebra. A tribute to the work of Robert Gilmer, pp. 1–20. Springer, New York (2006)
2. Badawi, A.: On ϕ-pseudo-valuation rings. Lect. Notes Pure Appl. Math. **205**, 101–110 (1999)
3. Badawi, A.: On divided commutative rings. Commun. Algebra. **27**, 1465–1474 (1999)
4. Badawi, A.: On ϕ-pseudo-valuation rings II. Houston J. Math. **26**, 473–480 (2000)
5. Badawi, A.: On ϕ-chained rings and ϕ-pseudo-valuation rings. Houston J. Math. **27**, 725–736 (2001)
6. Badawi, A.: On nonnil-Noetherian rings. Commun. Algebra **31**, 1669–1677 (2003)
7. Boisen, M.B.: Sheldon, P.B.: CPI-extension: overrings of integral domains with special prime spectrum. Can. J. Math. **29**, 722–737 (1977)
8. D'Anna, M.: A construction of Gorenstein rings. J. Algebra **306**(2), 507–519 (2006)
9. D'Anna, M.; Fontana, M.: The amalgamated duplication of a ring along a multiplicative-canonical ideal. Ark. Mat. **45**(2), 155–172 (2007)
10. D'Anna, M.; Fontana, M.: An amalgamated duplication of a ring along an ideal: the basic properties. J. Algebra Appl. **6**(3), 443–459 (2007)
11. D'Anna, M.; Finocchiaro, C.A.; Fontana, M.: Amalgamated algebras along an ideal. In: Commutative Algebra and its Applications, pp. 155–172. Walter De Gruyter, NY (2009)
12. D'Anna, M.; Finocchiaro, C.A.; Fontana, M.: Properties of chains of prime ideals in amalgamated algebras along an ideal. J. Pure Appl. Algebra **214**, 1633–1641 (2010)
13. Dobbs, D.E.: Divided rings and going-down. Pac. J. Math. **67**, 353–363 (1976)
14. Dorroh, J.L.: Concerning adjunctions to algebras. Bull. Am. Math. Soc. **38**, 85–88 (1932)
15. Fontana, M.: Topologically defined classes of commutative rings. Ann. Math. Pura Appl. **123**, 331–355 (1980)
16. Hizem, S.: Benhissi, A.: Nonnil-Noetherian rings and the SFT property. Rocky Mt. J. Math. **41**(5), 1483–1500 (2011)
17. Nagata, M.: Local Rings. Interscience, New York (1962)